Matrix Algebra

Questions

Q1.

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

(a) Show that 2 is a repeated eigenvalue of **A** and find the other eigenvalue.

(5)

(b) Hence find three non-parallel eigenvectors of **A**.

(4)

(c) Find a matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.

(2)

(Total for question = 11 marks)

Q2.

$$\mathbf{A} = \begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & p \\ -6 & 6 & -4 \end{pmatrix} \quad \text{where } p \text{ is a constant}$$

Given that $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ is an eigenvector for **A**

(a) (i) determine the eigenvalue corresponding to this eigenvector

(1)

(ii) hence show that $p = 2$

(2)

(iii) determine the remaining eigenvalues and corresponding eigenvectors of **A**

(7)

(b) Write down a matrix **P** and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{PDP}^{-1}$

(1)

(c) (i) Solve the differential equation $\dot{u} = ku$, where k is a constant.

(2)

With respect to a fixed origin *O*, the velocity of a particle moving through space is modelled by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

By considering $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so that $\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$

(ii) determine a general solution for the displacement of the particle.

(4)

(Total for question = 17 marks)

Q3.

Given that

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$$

(a) find the characteristic equation for the matrix **A**, simplifying your answer.

(2)

(b) Hence find an expression for the matrix $\mathbf{A}^{-1}$ in the form $\lambda\mathbf{A} + \mu\mathbf{I}$, where λ and μ are constants to be found.

(3)

(Total for question = 5 marks)

Q4.

Matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & a \\ -3 & b & 1 \\ 0 & 1 & a \end{pmatrix}$$

where *a* and *b* are integers, such that $a < b$

Given that the characteristic equation for **M** is

$$\lambda^3 - 7\lambda^2 + 13\lambda + c = 0$$

where *c* is a constant,

(a) determine the values of *a*, *b* and *c*.

(5)

(b) Hence, using the Cayley–Hamilton theorem, determine the matrix $\mathbf{M}^{-1}$

(3)

(Total for question = 8 marks)

Q5.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$

(7)

(Total for question = 7 marks)

Q6.

Given that

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$$

(a) find the characteristic equation of the matrix **A**.

(2)

(b) Hence show that $\mathbf{A}^3 = 43\mathbf{A} - 42\mathbf{I}$.

(3)

(Total for question = 5 marks)

Q7.

The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

(a) Show that 4 is an eigenvalue of **M**, and find the other two eigenvalues.

(4)

(b) For each of the eigenvalues find a corresponding eigenvector.

(4)

(c) Find a matrix **P** such that $\mathbf{P}^{-1}\mathbf{MP}$ is a diagonal matrix.

(2)

(Total for question = 10 marks)

Q8.

(i)

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$$

(a) Show that the characteristic equation for **A** is $\lambda^2 - 5\lambda + 6 = 0$

(2)

(b) Use the Cayley-Hamilton theorem to find integers *p* and *q* such that

$$\mathbf{A}^3 = p\mathbf{A} + q\mathbf{I}$$

(3)

(ii) Given that the 2 × 2 matrix **M** has eigenvalues –1 + i and –1 – i,

with eigenvectors $\begin{pmatrix} 1 \\ 2-\mathrm{i} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2+\mathrm{i} \end{pmatrix}$ respectively, find the matrix **M**.

(5)

(Total for question = 10 marks)

Q9.

$$\mathbf{M} = \begin{pmatrix} 1 & k & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

where *k* is a constant.

(a) Show that, in terms of *k*, a characteristic equation for **M** is given by

$$\lambda^3 - (2k + 13)\lambda + 5(k + 6) = 0$$

(3)

Given that det **M** = 5

(b) (i) find the value of *k*

(ii) use the Cayley-Hamilton theorem to find the inverse of **M**.

(7)

(Total for question = 10 marks)

Mark Scheme – Matrix Algebra

Q1.

Question	Scheme	Marks	AOs
(a)	$\lvert\mathbf{A}-\lambda\mathbf{I}\rvert=\begin{vmatrix}6-\lambda & -2 & 2\\ -2 & 3-\lambda & -1\\ 2 & -1 & 3-\lambda\end{vmatrix}=(6-\lambda)[...]-(-2)[...]+2[...]=...$	M1	1.1b
	$(6-\lambda)\left((3-\lambda)^2-1\right)+2\left(2(\lambda-3)+2\right)+2\left(2-2(3-\lambda)\right)(=0)$ $\left(\lambda^3-12\lambda^2+36\lambda-32=0\right)$	A1	1.1b
	$=(\lambda-2)(\lambda^2+...\lambda+...)$	M1	2.1
	$=(\lambda-2)(\lambda^2-10\lambda+16)=(\lambda-2)^2(\lambda-8)\Rightarrow\lambda=2$ is a repeated eigenvalue *	A1*	2.2a
	$\lambda=8$	B1	1.1b
		(5)	
(b)	$\begin{pmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{pmatrix}\mathbf{v}=2\begin{pmatrix}x\\ y\\ z\end{pmatrix}$ or $\begin{pmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{pmatrix}\mathbf{v}=8\begin{pmatrix}x\\ y\\ z\end{pmatrix}\Rightarrow\mathbf{v}=...$	M1	1.1b
	Obtains any multiple of $\begin{pmatrix}2\\ -1\\ 1\end{pmatrix}$ for $\lambda=8$	A1	1.1b
	Obtains any (non-zero) multiple or linear combination of $\begin{pmatrix}-1\\ 0\\ 2\end{pmatrix}$ or $\begin{pmatrix}1\\ 2\\ 0\end{pmatrix}$ or $\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}$ for $\lambda=2$	A1	1.1b
	Obtains a **different** linear combination or (non-zero) multiple of different vector from $\begin{pmatrix}-1\\ 0\\ 2\end{pmatrix}$ or $\begin{pmatrix}1\\ 2\\ 0\end{pmatrix}$ or $\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}$ for $\lambda=2$	A1	3.1a
		(4)	
(c)	Forms a matrix with their eigenvectors as columns	M1	1.2
	E.g. $\begin{pmatrix}-1 & 1 & 2\\ 0 & 2 & -1\\ 2 & 0 & 1\end{pmatrix}$	A1ft	1.1b
		(2)	
			(11 marks)

Notes
(a) M1: Attempts to expand the determinant to find the characteristic polynomial. Note: other methods of expanding the determinant are possible. If unsure send to review. A1: Correct expansion need not be simplified. (Need not see set equal to zero) Allow recovery of missing brackets if indicated by later working. M1: Attempts to take out a factor of $(\lambda - 2)$ of their equation (may first expand to cubic or may spot the factor and take out without full expansion). E.g $(6-\lambda)\left((3-\lambda)^2-1\right)+2\left(2(\lambda-3)+2\right)+2\left(2-2(3-\lambda)\right)=(6-\lambda)(4-\lambda)(2-\lambda)+4(\lambda-2)+4(\lambda-2)$ $=(\lambda-2)\left((6-\lambda)(4-\lambda)+4+4\right)$ This is for a method that will allow λ to be shown as a repeated eigenvalue, so just stating two solutions is not sufficient, factorisation must be seen. A1*: Obtains a correct factor of $(\lambda - 2)^2$ and deduces that 2 is a repeated eigenvalue. Must see **statement** about 2 being repeated. (Just listing 2 twice is not sufficient.) B1: (Note this is A1 on ePEN) Obtains and identifies 8 as the other eigenvalue (B0 if not identified in (a) but full marks can be scored in (b) and (c) for use of 8 as eigenvalue) (b) M1: Uses a correct method to find at least one eigenvector A1: Obtains one correct eigenvector for $\lambda = 8$ A1: Obtains one correct eigenvector for $\lambda = 2$ A1: Obtains two correct linearly independent eigenvectors for $\lambda = 2$ Note some other common eigenvectors for $\lambda = 2$ are $\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\3\\1\end{pmatrix}, \begin{pmatrix}2\\5\\1\end{pmatrix}, \begin{pmatrix}1\\4\\2\end{pmatrix}$ (c) M1: Forms a matrix with their three **different non-zero** eigenvectors as columns or with their normalised (or any scaled version) of their eigenvectors. A1ft: Correct matrix with the eigenvectors (normalised/scaled) as columns in any order (follow through their three **different** vectors which are **not multiples** of any other)

Q2.

Question	Scheme	Marks	AOs
(a)(i)	$\begin{pmatrix}5&-2&5\\0&3&p\\-6&6&-4\end{pmatrix}\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}-2\\3-2p\\2\end{pmatrix}=-1\times\begin{pmatrix}2\\2p-3\\-2\end{pmatrix}$		
	Corresponding eigenvalue is -1	B1	1.1b
		(1)	
(a)(ii)	$2p-3=1\Rightarrow p=...$	M1	1.1b
	$p=2$ *	A1*	1.1b
		(2)	
(a)(iii)	$\det\begin{pmatrix}5-\lambda&-2&5\\0&3-\lambda&p\\-6&6&-4-\lambda\end{pmatrix}=0$ $\Rightarrow(5-\lambda)((3-\lambda)(-4-\lambda)-12)-(-2)(12)+5(6(3-\lambda))=0$	M1	1.1b
	$\Rightarrow\lambda^3-4\lambda^2+\lambda+6=0$	A1	1.1b
	$(\Rightarrow(\lambda+1)(\lambda^2-5\lambda+6)=0\Rightarrow(\lambda+1)(\lambda-2)(\lambda-3)=0)$ Eigenvalues are (-1), 2 and 3	A1	1.1b
	Either $\left.\begin{matrix}3x-2y+5z=0\\y+2z=0\\-6x+6y-6z=0\end{matrix}\right\}$ or $\left.\begin{matrix}2x-2y+5z=0\\2z=0\\-6x+6y-7z=0\end{matrix}\right\}\Rightarrow x/y/z=...$	M1	2.1
	Either $k\begin{pmatrix}3\\2\\-1\end{pmatrix}$ (for $\lambda=2$) or $m\begin{pmatrix}1\\1\\0\end{pmatrix}$ (for $\lambda=3$)	A1	1.1b
	Both $\left.\begin{matrix}3x-2y+5z=0\\y+2z=0\\-6x+6y-6z=0\end{matrix}\right\}$ and $\left.\begin{matrix}2x-2y+5z=0\\2z=0\\-6x+6y-7z=0\end{matrix}\right\}\Rightarrow x/y/z=...$	M1	2.1
	Both $k\begin{pmatrix}3\\2\\-1\end{pmatrix}$ (for $\lambda=2$) and $m\begin{pmatrix}1\\1\\0\end{pmatrix}$ (for $\lambda=3$)	A1	1.1b
		(7)	

(b)	E.g. $\mathbf{P}=\begin{pmatrix}2&3&1\\1&2&1\\-2&-1&0\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}-1&0&0\\0&2&0\\0&0&3\end{pmatrix}$	B1ft	2.2a
		(1)	
(c)(i)	$\dot{u}=ku\Rightarrow\int\frac{1}{u}\mathrm{d}u=k\int\mathrm{d}t\Rightarrow\ln u=kt(+c)$	M1	1.1b

	So $u = Ae^{kt}$ or $u = e^{kt+c}$	A1	1.1b
		(2)	
(ii)	$\begin{pmatrix}\dot{x}\\\dot{y}\\\dot{z}\end{pmatrix} = \mathbf{PDP}^{-1}\begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow \begin{pmatrix}\dot{u}\\\dot{v}\\\dot{w}\end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix}\dot{x}\\\dot{y}\\\dot{z}\end{pmatrix} = \mathbf{D}\begin{pmatrix}u\\v\\w\end{pmatrix} = \begin{pmatrix}-u\\2v\\3w\end{pmatrix}$	M1	3.1b
	$\Rightarrow \begin{pmatrix}u\\v\\w\end{pmatrix} = \begin{pmatrix}Ae^{-t}\\Be^{2t}\\Ce^{3t}\end{pmatrix}$	M1	2.2a
	$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \mathbf{P}\begin{pmatrix}Ae^{-t}\\Be^{2t}\\Ce^{3t}\end{pmatrix} = \ldots$	M1	3.4
	$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2Ae^{-t}+3Be^{2t}+Ce^{3t}\\Ae^{-t}+2Be^{2t}+Ce^{3t}\\-2Ae^{-t}-Be^{2t}\end{pmatrix}$	A1	1.1b
		(4)	
			(17 marks)

Notes:
(a)(i) B1: For the correct eigenvalue of −1 (ii) M1: Correct equation with their eigenvalue set up – need only see middle equation for this. A1*: Correct proof (full matrix calculation not necessary). (iii) M1: Applies $\det(\mathbf{A}-\lambda\mathbf{I})=0$ to achieve a cubic in λ (or other variable, simplification not required). Allow with p used instead of 2, and look for two correct "terms" in the expansion leading to a cubic as evidence of the expansion. A1: Correct simplified cubic. Note this may be implied by correct answers from a calculator following a correct expansion seen for the M. A1: Correct eigenvalues M1: Forms and solves eigenvector equations for at least one (other than −1) eigenvalue. A1: One correct (other) eigenvector M1: Both eigenvectors attempted. A1: Both (other) eigenvectors correct.
(b) B1ft: A correct corresponding **P** and **D**, follow through on their answer to (a). Columns may be in different order, but should be consistent for their **P** and **D**.
(c)(i) M1: Separates variables and attempts the integration (constant not required). A1: Correct answer for $u = \ldots$, either form, including constant of integration

(ii) **NB different orderings of the columns of P and D will give the terms in different orders here.** **M1**: Uses their **P** and **D** to transform system into equation in u, v and w (may be implied). **M1**: Forms the solution for u, v and w using their eigenvalues. **M1**: Reverses the substitution (multiplies by their **P**) to get solution for x, y and z. **A1**: Correct answer, in matrix form or as separate equations – award when first seen and isw.

Q3.

Question	Scheme	Marks	AOs
(a)	$\det\begin{pmatrix}3-\lambda & 2\\ 2 & 2-\lambda\end{pmatrix} = (3-\lambda)(2-\lambda)-4(=0)$	M1	1.1b
	$\lambda^2-5\lambda+2=0$	A1	1.1b
		(2)	
(b)	$\mathbf{A}^2-5\mathbf{A}+2\mathbf{I}=0$	B1ft	1.1b
	Multiplies through by $\mathbf{A}^{-1}$ $\mathbf{A}-5\mathbf{I}+2\mathbf{A}^{-1}=0$ and rearranges to get $\mathbf{A}^{-1}=...$ **OR** Rearranges to make **I** the subject, takes out a factor of **A** and rearranges to get $\mathbf{A}^{-1}=...$ $\mathbf{I}=\frac{(5\mathbf{A}-\mathbf{A}^2)}{2}=\mathbf{A}\frac{(5\mathbf{I}-\mathbf{A})}{2}\Rightarrow \mathbf{A}^{-1}=...$ **OR** Rearranges to make **I** the subject and multiplies through by $\mathbf{A}^{-1}$ $\mathbf{I}=\frac{5}{2}\mathbf{A}-\frac{1}{2}\mathbf{A}^2\Rightarrow \mathbf{A}^{-1}=\frac{5}{2}\mathbf{A}\mathbf{A}^{-1}-\frac{1}{2}\mathbf{A}^2\mathbf{A}^{-1}$	M1	3.1a
	Identifies $\mathbf{A}^{-1}=-\frac{1}{2}\mathbf{A}+\frac{5}{2}\mathbf{I}$	A1	1.1b
		(3)	
			(5 marks)

Notes
(a) **M1**: Complete method to find the characteristic equation, condone missing = 0 **A1**: Obtains a correct three term quadratic equation – may use any variable.
(b) **B1ft**: Uses Cayley Hamilton Theorem to produce equation replacing λ with **A** and constant term with constant multiple of the identity matrix **I** **M1**: A complete method using part (a) to find $\mathbf{A}^{-1}$ Multiplies through by $\mathbf{A}^{-1}$ and rearranges to get $\mathbf{A}^{-1}=...$ Or rearranges to make **I** the subject, takes out a factor of **A**, and rearranges to get $\mathbf{A}^{-1}=...$ Or rearranges to make **I** the subject and multiplies through by $\mathbf{A}^{-1}$ to get $\mathbf{A}^{-1}=...$ **A1**: Correct expression for $\mathbf{A}^{-1}$, must be using their answer to part (a).

Q4.

Question	Scheme	Marks	AOs
(a)	$\begin{vmatrix} 1-\lambda & 0 & a \\ -3 & b-\lambda & 1 \\ 0 & 1 & a-\lambda \end{vmatrix}$ $= (1-\lambda)[(b-\lambda)(a-\lambda)-1] + a(-3)(=0)$	M1	1.1b
	$\lambda^3 - (a+b+1)\lambda^2 + (a+b+ab-1)\lambda + (3a+1-ab)(=0)$ o.e. $-\lambda^3 + (a+b+1)\lambda^2 - (a+b+ab-1)\lambda + (ab-3a-1)(=0)$ o.e.	A1	1.1b
	$\lambda^2 \Rightarrow\ a+b+1=7$ and $\lambda \Rightarrow\ a+b+ab-1=13$ Solves simultaneously e.g. $a+b=6, ab=8$ For example: leading to $a^2-6a+8=0 \Rightarrow a=...$	M1	3.1a
	$a=2, b=4$	A1	1.1b
	$c=-1$	A1	2.2a
		(5)	
(b)	$\mathbf{M}^3 - 7\mathbf{M}^2 + 13\mathbf{M} + \text{'their c'}\,\mathbf{I} = 0$	B1ft	1.1b
	$I = M^3 - 7M^2 + 13M \Rightarrow M^{-1} = M^2 - 7M + 13I$ $\Rightarrow M^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}^2 - 7\begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix} = ...$ $= \begin{pmatrix} 1 & 2 & 6 \\ -15 & 17 & 0 \\ -3 & 6 & 5 \end{pmatrix} - 7\begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix} = ...$	M1	3.1a
	$M^{-1} = \begin{pmatrix} 7 & 2 & -8 \\ 6 & 2 & -7 \\ -3 & -1 & 4 \end{pmatrix}$	A1	1.1b
		(3)	
			(8 marks)

Notes:
(a) **M1:** Correct method to find the characteristic equation for **M**, condone missing = 0, and one slip as long as the intention is clear **A1:** Multiplies out to achieve a correct characteristic equation, condone missing = 0 **M1:** A complete method to find the values of the constants a or b. Equates their coefficients for λ^2 and λ and solves simultaneously to find values for a or b. **A1:** Deduces the correct values for a and b. $(a < b)$ following correct simultaneous equations **A1:** Deduces the correct value for c.
(b) **B1ft:** Uses Cayley-Hamilton theorem to produce equation replacing λ with **M** and constant term with constant multiple of the identity matrix **I**. Follow through on their value for c. This mark may be implied by the M mark. **M1:** A complete method to find M^{-1} using the Cayley-Hamilton theorem. The minimum is for writing an expression for M^{-1} from their characteristic equation, for example $M^{-1} = M^2 - 7M + 13I$ and then stating an answer for M^{-1}, they may have used their calculator, there is no need to check. **A1:** Correct M^{-1}

Q5.

Question	Scheme	Marks	AOs
	$\|\mathbf{A}-\lambda\mathbf{I}\| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda)+2=0$	M1	3.1a
	$\lambda_1 = 2,\quad \lambda_2 = 3$	A1	1.1b
	$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$	M1	2.1
	$2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $3, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$	A1	1.1b
	$2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $3, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$	A1	1.1b
	$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$	B1ft	1.1b
	$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	B1ft	2.2a
		(7)	
			(7 marks)

Notes

M1: Correct strategy for finding eigenvalues
A1: Correct eigenvalues
M1: Uses at least one of their eigenvalues correctly to find a corresponding eigenvector
A1: One correct eigenvalue/eigenvector pair
A1: Both pairs correct
B1ft: Correct follow through **D** or **P** clearly identified as **D** or **P**
B1ft: **P** and **D** both correct and consistent and identified as **D** and **P**

Note that the correct matrices may be implied by e.g.

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

Q6.

Question	Scheme	Marks	AOs
(a)	Consider $\det\begin{pmatrix} 3-\lambda & 1 \\ 6 & 4-\lambda \end{pmatrix} = (3-\lambda)(4-\lambda)-6$	M1	1.1b
	So $\lambda^2 - 7\lambda + 6 = 0$ is characteristic equation	A1	1.1b
		(2)	
(b)	So $\mathbf{A}^2 = 7\mathbf{A} - 6\mathbf{I}$	B1ft	1.1b
	Multiplies both sides of their equation by **A** so $\mathbf{A}^3 = 7\mathbf{A}^2 - 6\mathbf{A}$	M1	3.1a
	Uses $\mathbf{A}^3 = 7(7\mathbf{A} - 6\mathbf{I}) - 6\mathbf{A}$ So $\mathbf{A}^3 = 43\mathbf{A} - 42\mathbf{I}$ *	A1*cso	1.1b
		(3)	
		(5 marks)	

Notes:

(a)
M1: Complete method to find characteristic equation
A1: Obtains a correct three term quadratic equation – may use variable other than λ

(b)
B1ft: Uses Cayley Hamilton Theorem to produce equation replacing λ with **A** and constant term with constant multiple of identity matrix, **I**
M1: Multiplies equation by **A**
A1*: Replaces $\mathbf{A}^2$ by linear expression in **A** and achieves printed answer with no errors

Q7.

Question	Scheme	Marks	AOs
(a)	Finds the characteristic equation $(2-\lambda)^2(4-\lambda)-(4-\lambda)=0$	M1	2.1
	so $(4-\lambda)(\lambda^2-4\lambda+3)=0$ so $\lambda=4$ *	A1*	2.2a
	Solves quadratic equation to give	M1	1.1b
	$\lambda=1$ and $\lambda=3$	A1	1.1b
		(4)	
(b)	Uses a correct method to find an eigenvector	M1	1.1b
	Obtains a vector parallel to one of $\begin{pmatrix}0\\0\\1\end{pmatrix}$ or $\begin{pmatrix}1\\1\\1\end{pmatrix}$ or $\begin{pmatrix}3\\-3\\1\end{pmatrix}$	A1	1.1b
	Obtains two correct vectors	A1	1.1b
	Obtains all three correct vectors	A1	1.1b
		(4)	
(c)	Uses their three vectors to form a matrix	M1	1.2
	$\begin{pmatrix}0&1&3\\0&1&-3\\1&1&1\end{pmatrix}$ or other correct answer with columns in a different order.	A1	1.1b
		(2)	
		(10 marks)	

Notes:
(a) **M1:** Attempts to find the characteristic equation (there may be one slip) **A1*:** Deduces that $\lambda=4$ is a solution by the method shown or by checking that $\lambda=4$ satisfies the characteristic equation **M1:** Solves their quadratic equation **A1:** Obtains the two correct answers as shown above
(b) **M1:** Uses a correct method to find an eigenvector **A1:** Obtains one correct vector (may be a multiple of the given vectors) **A1:** Obtains two correct vectors (may be multiples of the given vectors) **A1:** Obtains all three correct vectors (may be multiples of the given vectors)

Notes: (continued)
(c) **M1:** Forms a matrix with their vectors as columns **A1:** $\begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & -3 \\ 1 & 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 3 & 1 & 0 \\ -3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ or other correct alternative

Q8.

Question	Scheme	Marks	AOs
(i)(a)	$\begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)+2=0$	M1	1.1b
	$\Rightarrow 4-5\lambda+\lambda^2+2=0 \Rightarrow \lambda^2-5\lambda+6=0$*	A1*	1.1b
		(2)	
(b)	$\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I} = 0$	M1	1.1b
	$\mathbf{A}^3 - 5\mathbf{A}^2 + 6\mathbf{A} = 0 \Rightarrow \mathbf{A}^3 = 5(5\mathbf{A} - 6\mathbf{I}) - 6\mathbf{A}$	M1	3.1a
	$\mathbf{A}^3 = 19\mathbf{A} - 30\mathbf{I}$	A1	1.1b
		(3)	
	Alternative to part (b)		
	$\lambda^2-5\lambda+6=0 \Rightarrow \lambda^3-5\lambda^2+6\lambda=0 \Rightarrow \lambda^3=5(5\lambda-6)-6\lambda$	M1	3.1a
	$\mathbf{A}^3 = 5(5\mathbf{A} - 6\mathbf{I}) - 6\mathbf{A}$	M1	1.1b
	$\mathbf{A}^3 = 19\mathbf{A} - 30\mathbf{I}$	A1	1.1b
		(3)	

Question	Scheme	Marks	AOs
(ii)	$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2-\mathrm{i} \end{pmatrix} = (-1+\mathrm{i})\begin{pmatrix} 1 \\ 2-\mathrm{i} \end{pmatrix}$ or $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2+\mathrm{i} \end{pmatrix} = (-1-\mathrm{i})\begin{pmatrix} 1 \\ 2+\mathrm{i} \end{pmatrix}$	M1	1.1b
	$a+b(2-\mathrm{i})=-1+\mathrm{i} \quad a+b(2+\mathrm{i})=-1-\mathrm{i}$ $c+d(2-\mathrm{i})=-1+3\mathrm{i} \quad c+d(2+\mathrm{i})=-1-3\mathrm{i}$	A1	1.1b
	$a+b(2-\mathrm{i})=-1+\mathrm{i},\ a+b(2+\mathrm{i})=-1-\mathrm{i} \Rightarrow a=1,\ b=-1$ or $c+d(2-\mathrm{i})=-1+3\mathrm{i},\ c+d(2+\mathrm{i})=-1-3\mathrm{i} \Rightarrow c=5,\ d=-3$	M1 A1	3.1a 1.1b
	$\mathbf{M} = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}$	A1	2.2a
		(5)	
	Alternative to part (ii)		
	$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 2-\mathrm{i} & 2+\mathrm{i} \end{pmatrix} \Rightarrow \mathbf{P}^{-1} = \frac{1}{2\mathrm{i}}\begin{pmatrix} 2+\mathrm{i} & -1 \\ \mathrm{i}-2 & 1 \end{pmatrix}$	M1 A1	1.1b 1.1b
	$\mathbf{D} = \mathbf{P}^{-1}\mathbf{MP} \Rightarrow \mathbf{M} = \mathbf{PDP}^{-1}$ $\mathbf{M} = \frac{1}{2\mathrm{i}}\begin{pmatrix} 1 & 1 \\ 2-\mathrm{i} & 2+\mathrm{i} \end{pmatrix}\begin{pmatrix} -1+\mathrm{i} & 0 \\ 0 & -1-\mathrm{i} \end{pmatrix}\begin{pmatrix} 2+\mathrm{i} & -1 \\ \mathrm{i}-2 & 1 \end{pmatrix} = \ldots$	M1	3.1a
	$\mathbf{M} = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}$	A1 A1	1.1b 2.2a
		(5)	
			(10 marks)

Notes
(i)(a) M1: Attempts the determinant of $\mathbf{A} - \lambda\mathbf{I}$ A1*: Fully correct proof (i)(b) M1: Applies the Cayley-Hamilton theorem to the equation given in (a)(i) M1: A full method leading to $\mathbf{A}^3$ by multiplying by **A** and substituting for $\mathbf{A}^2$ A1: Deduces the correct expression or correct values for p and q **<u>Alternative</u>** M1: A full method leading to λ^3 in terms of λ M1: Applies the Cayley-Hamilton theorem A1: Deduces the correct expression or correct values for p and q (ii) M1: Uses a general matrix and sets up at least one matrix equation using the information given in the question A1: Correct equations in terms of a, b, c and d M1: Solves simultaneously to find values for all of a, b, c and d A1: One correct pair of values A1: Deduces the correct matrix **M** **<u>Alternative:</u>** M1: Attempts to find the inverse of the matrix of eigenvectors A1: Correct matrix M1: Attempts $\mathbf{PDP}^{-1}$ where **D** is the diagonal matrix of eigenvalues A1: At least 2 elements correct A1: Deduces the correct matrix **M**

Q9.

Question	Scheme	Marks	AOs
(a)	Sight of $\det(\mathbf{M}-\lambda\mathbf{I})=0$	B1	1.1a
	$\begin{vmatrix}1-\lambda & k & -2\\ 2 & -4-\lambda & 1\\ 1 & 2 & 3-\lambda\end{vmatrix}=0\Rightarrow$ $(1-\lambda)[(-4-\lambda)(3-\lambda)-2]-k[2(3-\lambda)-1]+(-2)[4-(-4-\lambda)]=0$	M1	1.1b
	$\Rightarrow(1-\lambda)(\lambda^2+\lambda-14)-k(5-2\lambda)-16-2\lambda=0$ $\Rightarrow\lambda^3-(2k+13)\lambda+5(k+6)=0$ *	A1*	2.1
		(3)	
(b)	(i) $\pm5(k+6)=5\Rightarrow k=\ldots$ or $(-12-2)-k(6-1)-2(4+4)=5\Rightarrow k=\ldots$	M1	1.1b
	$k=-7$	A1	2.2a
	(ii) Hence by the C-H theorem $\mathbf{M}^3+\mathbf{M}-5\mathbf{I}=\mathbf{0}$	M1	2.1
	Multiplying by $\mathbf{M}^{-1}$ gives $\mathbf{M}^2+\mathbf{I}-5\mathbf{M}^{-1}=\mathbf{0}\Rightarrow\mathbf{M}^{-1}=\ldots$	M1	3.1a
	So $\mathbf{M}^{-1}=\frac{1}{5}(\mathbf{M}^2+\mathbf{I})$	A1	1.1b
	$=\frac{1}{5}\left(\begin{pmatrix}-15 & 17 & -15\\ -5 & 4 & -5\\ 8 & -9 & 9\end{pmatrix}+\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)=\ldots$	M1	1.1b
	$=\frac{1}{5}\begin{pmatrix}-14 & 17 & -15\\ -5 & 5 & -5\\ 8 & -9 & 10\end{pmatrix}$ or $\begin{pmatrix}-\frac{14}{5} & \frac{17}{5} & -3\\ -1 & 1 & -1\\ \frac{8}{5} & -\frac{9}{5} & 2\end{pmatrix}$	A1	1.1b
		(7)	
		(10 marks)	

Notes:

(a)

B1: Recalls characteristic equation is found using $\det(\mathbf{M}-\lambda\mathbf{I})=0$

M1: Attempts to expand the determinant

A1*: Achieves the correct equation with no errors and at least one intermediate step following the expansion.

(b)(i)

M1: Attempts to use determinant equals 5 to find k. May be attempted by finding determinant from original matrix, or attempt at using the "$-5(k+6)$" from the expansion in (a) (allow $\pm$ for the method mark).

A1: $k=-7$

(ii)

M1: Attempts to use the Cayley-Hamilton theorem to set up a matrix equation. The equation should be correct for their k, including correct use of **I**.

M1: Realises the need to multiply the equation through (either side) by $\mathbf{M}^{-1}$ and rearrange to make $\mathbf{M}^{-1}$ the subject.

A1: $\mathbf{M}^{-1}=\frac{1}{5}(\mathbf{M}^2+\mathbf{I})$

M1: Proceeds to find $\mathbf{M}^{-1}$ from their equation.

A1: Correct answer.